18F03.

Lecture 8.

Adic spaces I.

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Huber rings and Tate rings.

Refs: Ranleya's notes on "Huber rings".

Wedhorn's notes on "Adic spaces".

Basic example.

$(K, |\cdot|)$ a non-arch. valued field.

$$K \xrightarrow{|\cdot|} \underset{\text{tot. ord group}}{\Gamma} \cup \{0\} \qquad |\cdot| = e^{-v}$$

Valuation topology. Basis at 0:

$$K_{<\gamma} = \{x \in K : x < \gamma\}. \qquad (\text{Could also use } \leq.)$$

If $\Gamma \subseteq \mathbb{R}_{>0}$, $(K, |\cdot|)$ is an example of everything.

A - top. ring.

Def. A subset $S \subseteq A$ is bounded if for every open nbd $0 \in U$, there exists another open $0 \in V$ s.t. $V \cdot S \subseteq U$.

Def. An elt $a \in A$ is powerbounded if $\{a^n : n \geq 0\}$ is a bounded subset of A. $\quad A^\circ$

Def. $a \in A$ is topologically nilpotent if $a^n \to 0$. $\quad A^{\circ\circ}$

Topologically nilpotent $\Rightarrow$ powerbounded. Def. A is uniform if A° is bounded in A.

If $|\cdot|$ is a rank 1 valuation on a field K,

$$K^\circ = R_{|\cdot|} = \{x \in K : |x| \le 1\}.$$

$$K^{\circ\circ} = m_{|\cdot|} = \{x \in K : |x| < 1\}.$$

This is uniform.

Def (R. Huber, 1993). A top. ring A is Huber (or f-adic) if there exists an open subring A_0 and a f.g. $I \subseteq A_0$ s.t. the top. on A_0 is the I-adic.

(A_0, I) is a ring, ideal of definition.

Rems ① For any $a \in A$, $\{a + I^n\}$ is a basis of nbds.

② I is usually not an ideal in A.

③ If A is Huber, $S \subseteq A$ is bounded iff $I^m S \subseteq I$ for some m.

Def. Let A be a Huber ring. A is Tate if it has a topologically nilpotent unit $g \in A$.

Ex. $K, |\cdot|$, non-archimedean valued field, rank 1. Or finite rank.

$g \in m_{|\cdot|}$.

Suppose A is Tate with ring of def ($I \subseteq A_0$) and a top. nilp. unit $g \in A$.
Then, $g^n \in A_0$ for some $n \gg 0$. Also top. nilp., also a unit.
Eventually in I too.

A Tate, (A_0, I) ring of def.

Lemma. If $g \in A_0$, then the top. on A_0 is g-adic.

proof. Since I is open, $g^n \in I$ for some n. So $(g)^n \subseteq I$ for that n. Since g is a unit in A, gA_0 is an open nbd of 0 in A_0. So, $I^m \subseteq gA_0$ for some m.

Rem. In this case, $A_0[\frac{1}{g}] = A$.

Now, just assume A is Huber.

Prop. $A_0 \subseteq A$ a subring. A_0 is open and bounded iff A_0 is a ring of def.

proof. $(\Uparrow)$ Open clear. $I \subseteq A_0$ ideal of def. Then, $I \cdot A_0 \subseteq I$, so A_0 is bounded.

$(\Downarrow)$ A_0 open and bounded.
Let (B, I) be a pair of def. for A.
$I^m \subseteq A_0$ for some m.
$\uparrow$
Just a subgroup.

Since A_0 is bounded, there exists n s.t.

$$I^n A_0 \subseteq I^m.$$

$I^n = (x_1, \dots, x_r)$ in B, set

$$J = \sum x_i A_0.$$

Now, need to know that the induced topology is J-adic.

Have $J \subseteq I^n$.

$$I^{m+n} = \underbrace{I^m \cdot (Bx_1 + \dots + Bx_r)}_{= I^m x_1 + \dots + I^m x_r} \subseteq A_0 x_1 + \dots + A_0 x_r = J.$$

Prop. A° = union of all rings of definition.

If A is uniform, then A° is a ring of definition.

Maps. In general, a continuous homomorphism of Huber rings need not be bounded.

Ex. $\underset{\text{discrete}}{\mathbb{Q}_p} \xrightarrow{\ id\ } \underset{p\text{-adic}}{\mathbb{Q}_p}$

$A \longrightarrow B$ continuous map of Huber rings.

If A is Tate, so is B.

Def. A homomorphism $A \xrightarrow{f} B$ of Huber rings is adic if there exists rings of def $A_0 \subseteq A$, $B_0 \subseteq B$ and $I \subseteq A_0$ an IoD s.t. $f(A_0) \subseteq B_0$, $f(I) \cdot B_0$ is an IoD of B.

Turns out to be independent of I.

Takes bounded subsets to bounded subsets.